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6-j for mixed symmetry triads in K_{20}

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Abstract. We discuss the possible choices of conjugation and permutation matrices for mixed symmetry triads. This requires a short review of the previous choices for all triads. The primitive 6-j of the finite group K_{20} are then calculated as an example of a group that contains mixed symmetry triads.

1. Introduction

Although numerous 6-j have been calculated for various groups, relatively few workers have attempted to calculate 6-j with multiplicity and very few (see Zhang and Xiangzhu 1987, Gao and Chen 1985) have attempted to calculate 6-j involving mixed symmetry couplings. Since mixed symmetry triads are a common occurrence in most Lie groups (exceptions are SO_2, SO_3 and SU_3) and in the symmetric groups S_n, for $n \geqslant 6$, these cases must eventually be analysed.

In our development of a PASCAL program to perform the calculation of 6-j for a general compact group, it was necessary to design it to handle mixed symmetry triads. The program only has the selection rules for the group as information and also assumes a particular choice of permutation and conjugation matrices for the various triads. Therefore we need to study the available choices for these matrices, how the various choices are interrelated and which choice is the most convenient form from the viewpoint of both group theory and programming.

We study the various phase choices and decide on an appropriate choice for mixed symmetry triads and then check our results by calculating primitive 6-j for a small finite group with mixed symmetry triads. Bickerstaff suggested that we try the K-metacyclic group of order 20 (K_{20}) as a test for our ideas since it was a group he had attempted but not completed due to the effort required to solve the large number of 6-j containing mixed symmetry. All but one of the irreps in K_{20} are one dimensional, but the non-trivial irrep has a mixed symmetry triad. The rather special nature of K_{20} also gives us an opportunity to produce an example of the calculation of core 6-j, as discussed in a previous paper (Searle and Butler 1988).

In § 2 we look at the general case and review what is known about the various matrices and the choices that have previously been used. We discuss the possible choices for the symmetry relations of mixed symmetry triads and consider their various merits in § 3. Finally in § 4 we introduce the group K_{20} and present our results for its primitive 6-j. These are the first 6-j we have found that are strictly complex, although such cases are not unknown for the 3-jm symbols (such as T to D_2; see Butler (1981)).

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2. Review

Derome and Sharp (1965) introduced unitary A, M and U matrices to describe the symmetries of a generalised 3-*jm* or 6-*j* symbol for any compact group. We will review these and other results for the choice of these matrices (see Butler 1975, Bickerstaff 1981) in this section. These transformation matrices are defined with respect to their effect on the group triad of a 3-*jm* (and hence four are required for a 6-*j* since it can be written as a product of four 3-*jm*).

We describe the U matrix first. This matrix relates 3-*jm* symbols with alternative coupling multiplicity choices via a unitary transformation

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ i_1 & i_2 & i_3 \end{pmatrix}^r_{\text{alt}} = \sum_{r'} U(\lambda_1\lambda_2\lambda_3)^r_{r'} \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ i_1 & i_2 & i_3 \end{pmatrix}^{r'}. \tag{2.1}$$

The U matrix therefore describes the freedom of choice we have in the value of the 3-*jm* or 6-*j* due to the coupling process (see Searle and Butler 1988), as distinct from the freedom in 3-*jm* symbols due to the freedom in branching multiplicity. The freedom described by the U matrix can be used to study the possible choices for the two matrices, A and M.

The M matrix gives the property of a 3-*jm* under a column permutation (a reordering of the coupling) in the following manner:

$$\begin{pmatrix} \lambda_a & \lambda_b & \lambda_c \\ i_a & i_b & i_c \end{pmatrix}^r = \sum_{r'} M(\pi, \lambda_1\lambda_2\lambda_3)^r_{r'} \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ i_1 & i_2 & i_3 \end{pmatrix}^{r'} \tag{2.2}$$

where π is the permutation performed on the indices (namely $(abc) = \pi(123)$). The elements of M are known as 3-*j* phases. Alternative choices of the permutation matrices are related via the U matrix as follows:

$$M'(\pi, \lambda_1\lambda_2\lambda_3) = U(\lambda_a\lambda_b\lambda_c)^\dagger M(\pi, \lambda_1\lambda_2\lambda_3) U(\lambda_1\lambda_2\lambda_3). \tag{2.3}$$

Derome (1966) has discussed the choices for the M matrix allowed by (2.3) and has found the simplest values of M that could be used. Whenever all three irreps of the triad $\lambda_1\lambda_2\lambda_3$ are not identical, M can be chosen as a diagonal matrix with diagonal entries of ± 1 by choosing $U(\lambda_a\lambda_b\lambda_c)$ in relation to $U(\lambda_1\lambda_2\lambda_3)$. In the case that $\lambda_1 = \lambda_2 = \lambda_3$ the M matrix can be chosen block diagonal with respect to symmetry type. However, three symmetry types may occur: these are the symmetric, mixed and antisymmetric types whose occurrence is related, respectively, to the occurrence of the scalar in the symmetric, mixed and antisymmetric parts of the cube of λ_1. For the symmetric and antisymmetric parts of the cube of λ_1 ($\lambda_1 \otimes \{3\}$ and $\lambda_1 \otimes \{1^3\}$, respectively) the blocks may be chosen as $I_{[3]}$ or $-I_{[1^3]}$. The dimension of the unit matrix is the same as the multiplicity of the scalar in the appropriate part of the cube. An occurrence of the scalar in $\lambda \otimes \{21\}$ means we have an occurrence of a so-called mixed symmetry triad, where the column permutation symmetry must be represented by two-dimensional matrices of the irrep [21] of S_3. Our M matrix for $\lambda_1 = \lambda_2 = \lambda_3\ (=\lambda)$ is then of the form

$$M(\pi, \lambda\lambda\lambda) = \begin{pmatrix} I_{[3]} & & \\ & M_{[21]} & \\ & & -I_{[1^3]} \end{pmatrix}$$

where the $M_{[21]}$ block is itself composed of the two-dimensional S_3 irrep matrix (for the permutation π) as blocks on its diagonal. $M_{[21]}$ has dimension of twice the multiplicity of the scalar in $\lambda \otimes \{21\}$.

The final unitary matrix to consider is the conjugation or A matrix, where

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ i_1 & i_2 & i_3 \end{pmatrix}^{r*} = \sum_{r' i_1' i_2' i_3'} A(\lambda_1\lambda_2\lambda_3)_{rr'} (\lambda_1)^{i_1 i_1'} (\lambda_2)^{i_2 i_2'} (\lambda_3)^{i_3 i_3'} \begin{pmatrix} \lambda_1^* & \lambda_2^* & \lambda_3^* \\ i_1' & i_2' & i_3' \end{pmatrix}^{r'}. \tag{2.4}$$

It has been usual to choose the A matrix equal to I for all couplings, although sometimes this choice does not give real coupling symbols (Sullivan 1983). The various forms of the A matrix are related to the other matrices via

$$A'(\lambda_1^*\lambda_2^*\lambda_3^*) = U(\lambda_1\lambda_2\lambda_3)^{\mathrm{T}} A(\lambda_1^*\lambda_2^*\lambda_3^*) U(\lambda_1^*\lambda_2^*\lambda_3^*) \tag{2.5}$$

and

$$A'(\lambda_a\lambda_b\lambda_c) = M(\pi^{-1}\lambda_a\lambda_b\lambda_c)^{\mathrm{T}} A(\lambda_1\lambda_2\lambda_3) M(\pi\lambda_1\lambda_2\lambda_3)^{\dagger}. \tag{2.6}$$

We shall discuss further the choices of A for mixed symmetry triads in the next section.

3. Choices of the S_3 irrep matrices

The usual matrix form for the two-dimensional representation of S_3 consists of real orthogonal matrices with the generator matrices $(12) = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$, $(123) = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}$. As a result, for a mixed symmetry triad pair $\lambda\lambda\lambda\, 1$ and $\lambda\lambda\lambda\, 2$, the 3-*jm* with permuted columns is a linear combination of 3-*jm* with unpermuted columns. In particular we have

$$\begin{pmatrix} \lambda & \lambda & \lambda \\ k & i & j \end{pmatrix}^1 = -\frac{1}{2}\begin{pmatrix} \lambda & \lambda & \lambda \\ i & j & k \end{pmatrix}^1 + \frac{\sqrt{3}}{2}\begin{pmatrix} \lambda & \lambda & \lambda \\ i & j & k \end{pmatrix}^2. \tag{3.1}$$

An alternative representation of this irrep has the second of these generators (the 3-cycle) diagonalised, giving $(12) = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$ and $(123) = \begin{pmatrix} \omega^2 & \\ & \omega \end{pmatrix}$ with $\omega = \exp(2\pi \mathrm{i}/3)$. Use of this matrix irrep would imply the use of complex 3-*jm* symbols, but the 3-*jm* with permuted columns is simply related to an unpermuted symbol, e.g.

$$\begin{pmatrix} \lambda & \lambda & \lambda \\ i & j & k \end{pmatrix}^1 = \begin{pmatrix} \lambda & \lambda & \lambda \\ j & i & k \end{pmatrix}^2 = \omega^2 \begin{pmatrix} \lambda & \lambda & \lambda \\ j & k & i \end{pmatrix}^1. \tag{3.2}$$

We wish to know whether there is any restriction on the use of either of these two choices and what effect they have on the choice of the A matrix.

At this stage we impose the requirement that the product of two orthogonal or two symplectic irreps contains only orthogonal irreps. This is a restriction that is satisfied by all triads of all the classical Lie groups, point groups, symmetric groups and many finite groups. With this restriction it has been shown (Butler 1975) that the A matrix is a symmetric unitary matrix which is block diagonal on symmetry type. We would like to be able to choose the A matrix to be the identity, as this is consistent with most previous workers' choices.

For the choices of the M matrix in § 2 it is known that we may choose $A = I$. If the real choice of the M matrix for a mixed symmetry triad is used then (2.6) shows that the A matrix must be a multiple of I. However, when we apply the generators of the complex choice of the M matrix to A we find

$$A(\lambda\lambda\lambda) = \begin{pmatrix} \omega & \\ & \omega^2 \end{pmatrix} A(\lambda\lambda\lambda) \begin{pmatrix} \omega & \\ & \omega^2 \end{pmatrix} \tag{3.3}$$

which requires that the diagonal elements of A are zero. The other generator then fixes A as a multiple of $\left({}_1 \; {}^1\right)$. This skew-diagonal matrix relates one conjugated symbol to the symbol for the other multiplicity of the pair, e.g.

$$\begin{pmatrix} \lambda & \lambda & \lambda \\ j & i & k \end{pmatrix}^{1^*} = \sum_{i',j',k'} (\lambda)^{ii'}(\lambda)^{jj'}(\lambda)^{kk'} \begin{pmatrix} \lambda^* & \lambda^* & \lambda^* \\ i' & j' & k' \end{pmatrix}^2. \tag{3.4}$$

We will choose $A = I$ and will choose the real set of permutation matrices. This will prevent the permuted forms of a symbol being complex if the symbol can be chosen real and allows the A matrix to be omitted in most applications of the Racah–Wigner algebra.

4. Primitive 6-*j* for K_{20}

To test the calculation of mixed symmetry triads using the above results we looked for a small finite group that contained such a triad. The K metacyclic group of order 20 (see Biedenharn *et al* 1968, Bovier *et al* 1981) is the smallest such group. It has only one irrep that is not one dimensional and all irreps are quasi-orthogonal. All the primitive 6-*j* that occur are core (as defined by Searle and Butler 1988) and the irrep with the mixed symmetry triad is also the primitive irrep (see tables 1 and 2) where the triads 4441 and 4442 form the only mixed symmetry pair.

The one basis 6-*j* that does not contain 444*r* is $\left\{\begin{smallmatrix} 2 & 2 & 1 \\ 4 & 4 & 4 \end{smallmatrix}\right\}$ and is trivially solved by using the normality relation. Those few non-basis 6-*j* that do not contain any of the triads 444*r* (see table 2) can then be readily solved by the Racah backcoupling equation.

Table 1. Character table.

	Class				
Irrep	E	T	S	T^2	T^3
0	1	1	1	1	1
1	1	−1	1	1	−1
2	1	i	1	−1	−i
3	1	−i	1	−1	i
4	4	0	−1	0	0

Table 2. Table of 3-*j*.

0 0 0 0+	4 4 0 0+	4 4 4 0+
4 4 4 1+	4 4 4 2−	1 1 0 0+
1 4 4 0+	2 4 4 0−	2 2 1 0+
	3 2 0 0+	

Solving for the large number of remaining 6-*j* is more complicated since the column permutation of any 6-*j* is related to a linear combination of some others (even when a permutation does not seem to alter the irreps, it will still permute the multiplicity indices).

For example, a (23) interchange of $\left\{\begin{smallmatrix}4&4&4\\4&4&4\end{smallmatrix}\right\}_{0011}$ is $\left\{\begin{smallmatrix}4&4&4\\4&4&4\end{smallmatrix}\right\}_{0101}$ which, by the symmetry relations, is equal to

$$\sum_{ab} M(23,444)_{aa'} M(23,444)_{bb'} \begin{Bmatrix}4&4&4\\4&4&4\end{Bmatrix}_{00a'b'}$$

since $M(23,444)_{0a}=\delta_{0a}$.

By making use of the results of Newmarch (1983) to find out which 6-*j* should be considered independent, of which there were 14, we were able to solve the independent set. We chose

$$\begin{Bmatrix}4&4&4\\4&4&4\end{Bmatrix}_{0000} \begin{Bmatrix}4&4&4\\4&4&4\end{Bmatrix}_{0011} \begin{Bmatrix}4&4&4\\4&4&4\end{Bmatrix}_{0012} \begin{Bmatrix}4&4&4\\4&4&4\end{Bmatrix}_{0111}$$
$$\begin{Bmatrix}4&4&4\\4&4&4\end{Bmatrix}_{0112} \begin{Bmatrix}4&4&4\\4&4&4\end{Bmatrix}_{1111} \begin{Bmatrix}1&4&4\\4&4&4\end{Bmatrix}_{0000} \begin{Bmatrix}1&4&4\\4&4&4\end{Bmatrix}_{0010}$$
$$\begin{Bmatrix}1&4&4\\4&4&4\end{Bmatrix}_{0110} \begin{Bmatrix}1&4&4\\4&4&4\end{Bmatrix}_{0120} \begin{Bmatrix}2&4&4\\4&4&4\end{Bmatrix}_{0000} \begin{Bmatrix}2&4&4\\4&4&4\end{Bmatrix}_{0010}$$
$$\begin{Bmatrix}2&4&4\\4&4&4\end{Bmatrix}_{0110} \begin{Bmatrix}2&4&4\\4&4&4\end{Bmatrix}_{0120}$$

as our independent set. By using both the orthogonality and Racah backcoupling equations (with normality for the few basis 6-*j*) we were able to obtain sufficient independent equations to resolve the remaining 6-*j*. The Racah backcoupling equations gave the necessary extra information because they include information on the symmetry of the triads. The simultaneous equations for the independent 6-*j* were created and solved by the algebraic program REDUCE, as were the symmetry relations for the related 6-*j*. The results were put into the Biedenharn-Elliott equation as an independent check that we had correctly solved the various equations. The set of 6-*j* are given in table 3, in the same format as used for the tables of Butler (1981).

The 6-*j* for this group, K_{20}, are the first set of 6-*j* to be calculated where the 6-*j* are strictly complex. It is impossible to find a different U matrix that will give pure real or imaginary values without producing imaginary values for some of the real 6-*j*. A change of multiplicity separation for the mixed symmetry pair affects 3-*jm* as in (2.1) and will affect 6-*j* similarly, except that there are four matrices, one for the multiplicity of each triad (see equation (4.1) of Searle and Butler (1988)). Any such change merely moves the pure real, pure imaginary and strictly complex values around table 3.

5. Conclusion

This paper has reviewed some of the knowledge about the symmetry matrices for the generalised 3-*jm* and 6-*j* symbols. We have then discussed the possible choices for the complex conjugation matrix given various choices of permutation matrix for a mixed symmetry triad. These choices have then been used in calculating the primitive 6-*j* of the finite group, K_{20}.

Table 3. Table of 6-*j*.

0 0 0

0 0 0	0000	+1

4 4 0

0 0 4	0000	+1/2
4 4 0	0000	+1/4

4 4 4

4 4 0	0000	+1/4
4 4 0	0001	0
4 4 0	0002	0
4 4 0	0011	+1/4
4 4 0	0012	0
4 4 0	0022	−1/4
4 4 4	0000	+1/6
4 4 4	0001	0
4 4 4	0002	0
4 4 4	0010	0
4 4 4	0011	−1/12
4 4 4	0012	0
4 4 4	0020	0
4 4 4	0021	0
4 4 4	0022	+1/12
4 4 4	0100	0
4 4 4	0101	+1/24
4 4 4	0102	$+1/8\sqrt{3}$
4 4 4	0110	+1/24
4 4 4	0111	$+1/12\sqrt{2}$
4 4 4	0112	$-1/4\sqrt{6}$
4 4 4	0120	$-1/8\sqrt{3}$
4 4 4	0121	$+1/4\sqrt{6}$
4 4 4	0122	$+1/12\sqrt{2}$
4 4 4	0200	0
4 4 4	0201	$+1/8\sqrt{3}$
4 4 4	0202	−1/24
4 4 4	0210	$-1/8\sqrt{3}$
4 4 4	0211	0
4 4 4	0212	$-1/6\sqrt{2}$
4 4 4	0220	−1/24
4 4 4	0221	$-1/6\sqrt{2}$
4 4 4	0222	0
4 4 4	1000	0
4 4 4	1001	+1/24
4 4 4	1002	$-1/8\sqrt{3}$
4 4 4	1010	+1/24
4 4 4	1011	$+1/12\sqrt{2}$
4 4 4	1012	$+1/4\sqrt{6}$
4 4 4	1020	$+1/8\sqrt{3}$
4 4 4	1021	$-1/4\sqrt{6}$
4 4 4	1022	$+1/12\sqrt{2}$
4 4 4	1100	−1/12
4 4 4	1101	$+1/12\sqrt{2}$
4 4 4	1102	0
4 4 4	1110	$+1/12\sqrt{2}$
4 4 4	1111	+1/8
4 4 4	1112	0
4 4 4	1120	0
4 4 4	1121	0

4 4 4 (continued)

4 4 4	1122	+1/24
4 4 4	1200	0
4 4 4	1201	$-1/4\sqrt{6}$
4 4 4	1202	$-1/6\sqrt{2}$
4 4 4	1210	$+1/4\sqrt{6}$
4 4 4	1211	0
4 4 4	1212	+1/24
4 4 4	1220	$-1/6\sqrt{2}$
4 4 4	1221	+1/24
4 4 4	1222	0
4 4 4	2000	0
4 4 4	2001	$-1/8\sqrt{3}$
4 4 4	2002	−1/24
4 4 4	2010	$+1/8\sqrt{3}$
4 4 4	2011	0
4 4 4	2012	$-1/6\sqrt{2}$
4 4 4	2020	−1/24
4 4 4	2021	$-1/6\sqrt{2}$
4 4 4	2022	0
4 4 4	2100	0
4 4 4	2101	$+1/4\sqrt{6}$
4 4 4	2102	$-1/6\sqrt{2}$
4 4 4	2110	$-1/4\sqrt{6}$
4 4 4	2111	0
4 4 4	2112	+1/24
4 4 4	2120	$-1/6\sqrt{2}$
4 4 4	2121	+1/24
4 4 4	2122	0
4 4 4	2200	+1/12
4 4 4	2201	$+1/12\sqrt{2}$
4 4 4	2202	0
4 4 4	2210	$+1/12\sqrt{2}$
4 4 4	2211	+1/24
4 4 4	2212	0
4 4 4	2220	0
4 4 4	2221	0
4 4 4	2222	+1/8
1 4 4	0000	−1/12
1 4 4	0001	$+1/6\sqrt{2}$
1 4 4	0002	$+1/2\sqrt{6}$
1 4 4	1000	$+1/6\sqrt{2}$
1 4 4	1001	+5/24
1 4 4	1002	$-1/8\sqrt{3}$
1 4 4	2000	$+1/2\sqrt{6}$
1 4 4	2001	$-1/8\sqrt{3}$
1 4 4	2002	+1/8
4 1 4	0000	−1/12
4 1 4	0001	$+1/6\sqrt{2}$
4 1 4	0002	$-1/2\sqrt{6}$
4 1 4	0100	$+1/6\sqrt{2}$
4 1 4	0101	+5/24
4 1 4	0102	$+1/8\sqrt{3}$
4 1 4	0200	$-1/2\sqrt{6}$
4 1 4	0201	$+1/8\sqrt{3}$
4 1 4	0202	+1/8
4 4 1	0000	−1/12
4 4 1	0001	$-1/3\sqrt{2}$

4 4 4 (continued)

4 4 1	0002	0
4 4 1	0010	$-1/3\sqrt{2}$
4 4 1	0011	+1/12
4 4 1	0012	0
4 4 1	0020	0
4 4 1	0021	0
4 4 1	0022	+1/4
2 4 4	0000	+1/12
2 4 4	0001	$(1+3i)/12\sqrt{2}$
2 4 4	0002	$(1-i)/4\sqrt{6}$
2 4 4	1000	$(1-3i)/12\sqrt{2}$
2 4 4	1001	+1/24
2 4 4	1002	$(1+2i)/8\sqrt{3}$
2 4 4	2000	$(1+i)/4\sqrt{6}$
2 4 4	2001	$(1-2i)/8\sqrt{3}$
2 4 4	2002	+1/8
4 2 4	0000	+1/12
4 2 4	0001	$(1-3i)/12\sqrt{2}$
4 2 4	0002	$-(1+i)/4\sqrt{6}$
4 2 4	0100	$(1+3i)/12\sqrt{2}$
4 2 4	0101	+1/24
4 2 4	0102	$(-1+2i)/8\sqrt{3}$
4 2 4	0200	$(-1+i)/4\sqrt{6}$
4 2 4	0201	$-(1+2i)/8\sqrt{3}$
4 2 4	0202	+1/8
4 4 2	0000	+1/12
4 4 2	0001	$-1/6\sqrt{2}$
4 4 2	0002	$+i/2\sqrt{6}$
4 4 2	0010	$-1/6\sqrt{2}$
4 4 2	0011	+1/6
4 4 2	0012	$+i/4\sqrt{3}$
4 4 2	0020	$-i/2\sqrt{6}$
4 4 2	0021	$-i/4\sqrt{3}$
4 4 2	0022	0

4 1 4

4 4 4	0000	−1/12
4 4 4	0010	$+1/6\sqrt{2}$
4 4 4	0020	$-1/2\sqrt{6}$
4 4 4	1000	$+1/6\sqrt{2}$
4 4 4	1010	+5/24
4 4 4	1020	$+1/8\sqrt{3}$
4 4 4	2000	$-1/2\sqrt{6}$
4 4 4	2010	$+1/8\sqrt{3}$
4 4 4	2020	+1/8

4 2 4

4 4 4	0000	+1/12
4 4 4	0010	$(1+3i)/12\sqrt{2}$
4 4 4	0020	$(-1+i)/4\sqrt{6}$
4 4 4	1000	$(1-3i)/12\sqrt{2}$
4 4 4	1010	+1/24
4 4 4	1020	$-(1+2i)/8\sqrt{3}$
4 4 4	2000	$-(1+i)/4\sqrt{6}$
4 4 4	2010	$(-1+2i)/8\sqrt{3}$
4 4 4	2020	+1/8

Table 3. (continued)

4 4 1	1 4 4	2 4 4 (continued)
4 4 4 0000 $-1/12$	0 4 4 0000 $+1/4$	4 4 4 0010 $(1+3i)/12\sqrt{2}$
4 4 4 0100 $-1/3\sqrt{2}$	4 1 0 0000 $+1/2$	4 4 4 0020 $(1-i)/4\sqrt{6}$
4 4 4 0200 0	4 4 4 0000 $-1/12$	4 4 4 0100 $(1+3i)/12\sqrt{2}$
4 4 4 1000 $-1/3\sqrt{2}$	4 4 4 0010 $+1/6\sqrt{2}$	4 4 4 0110 $+1/24$
4 4 4 1100 $+1/12$	4 4 4 0020 $+1/2\sqrt{6}$	4 4 4 0120 $(1+2i)/8\sqrt{3}$
4 4 4 1200 0	4 4 4 0100 $+1/6\sqrt{2}$	4 4 4 0200 $(1+i)/4\sqrt{6}$
4 4 4 2000 0	4 4 4 0110 $+5/24$	4 4 4 0210 $(1-2i)/8\sqrt{3}$
4 4 4 2100 0	4 4 4 0120 $-1/8\sqrt{3}$	4 4 4 0220 $+1/8$
4 4 4 2200 $+1/4$	4 4 4 0200 $+1/2\sqrt{6}$	1 4 4 0000 $-1/4$
	4 4 4 0210 $-1/8\sqrt{3}$	2 4 4 0000 $+1/4$
	4 4 4 0220 $+1/8$	3 4 4 0000 $+1/4$
4 4 2	1 4 4 0000 $+1/4$	
4 4 4 0000 $+1/12$		2 2 1
4 4 4 0100 $-1/6\sqrt{2}$	1 1 0	0 1 2 0000 $+1$
4 4 4 0200 $+i/2\sqrt{6}$	0 0 1 0000 $+1$	4 4 4 0000 $+1/2$
4 4 4 1000 $-1/6\sqrt{2}$	1 1 0 0000 $+1$	2 3 1 0000 $+1$
4 4 4 1100 $+1/6$		3 2 0 0000 $+1$
4 4 4 1200 $+i/4\sqrt{3}$	2 4 4	
4 4 4 2000 $-i/2\sqrt{6}$	0 4 4 0000 $-1/4$	3 2 0
4 4 4 2100 $-i/4\sqrt{3}$	4 2 0 0000 $-1/2$	0 0 2 0000 $+1$
4 4 4 2200 0	4 4 4 0000 $+1/12$	3 3 0 0000 $+1$

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